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Discrete Fourier–Neumann series

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Abstract

Let J_μ denote the Bessel function of order μ. The system

$$j_n^\alpha = \{j_n^\alpha(s)\}_{s\geqslant 1} = \left\{2\sqrt{\alpha+2n+1}\,\frac{J_{\alpha+2n+1}(p_s)}{ap_s|J_{\alpha+1}(ap_s)|}\right\}_{s\geqslant 1}$$

with $n = 0, 1, \ldots, \alpha > -1$, and where p_s denotes the sth positive zero of $J_\alpha(ax)$, is orthonormal in $l^2(\mathbb{N})$. In this paper, we study the mean convergence of the Fourier series with respect to this system. Also, we describe the space in which the span of the system is dense.

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1. Introduction

Let p_s be the sth positive zero of $J_\alpha(ax)$, where J_μ denotes the Bessel function of order μ and $a > 1$. In [9, Section 6.6, Lemma 2], it is proved that if m, n are positive integers or zero, $\alpha > -m-n-\gamma-1$ and not a negative integer,

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and $\gamma + m + n > -1$, then

$$\sum_{s=1}^{\infty} \frac{J_{\alpha+\gamma+2n+1}(p_s)J_{\alpha+\gamma+2m+1}(p_s)}{(ap_sJ_{\alpha+1}(ap_s))^2} = \frac{\delta_{n,m}}{4(\alpha+\gamma+2n+1)} + (-1)^{n+m}\frac{\sin(\gamma\pi)}{\pi}\int_0^{\infty}\frac{K_\alpha(t)}{tI_\alpha(t)} \times I_{\alpha+\gamma+2n+1}\left(\frac{t}{a}\right)I_{\alpha+\gamma+2m+1}\left(\frac{t}{a}\right)dt$$

with K_α the modified Hankel function and I_μ the modified Bessel function of the first kind. The previous formula, being $\gamma = 0$, provides an orthonormal system in $l^2(\mathbb{N})$ given, for $\alpha > -1$, by the sequences

$$j_n^\alpha = \{j_n^\alpha(s)\}_{s\geqslant 1} = \left\{2\sqrt{\alpha+2n+1}\,\frac{J_{\alpha+2n+1}(p_s)}{ap_s|J_{\alpha+1}(ap_s)|}\right\}_{s\geqslant 1}, \quad n = 0, 1, \dots .$$

We consider the partial sums of the Fourier series with respect to the system $\{j_n^\alpha\}_{n\geqslant 0}$

$$S_n(b,j) = \sum_{k=0}^{n} c_k(b)j_k^\alpha(j), \quad c_k(b) = \sum_{s=1}^{\infty} b(s)j_k^\alpha(s).$$

These series are the discrete analogous of Fourier–Neumann series (see [3,10]), so we will refer to S_nb as the discrete Fourier–Neumann series.

Fourier–Neumann series have had a prominent role in the study of band-limited functions for the Fourier transform (see [1]) and in the analysis of dual integral equations (see [4]). Moreover, some of the operators appearing in Fourier–Neumann expansions are related to the disc multiplier for the Fourier transform (see [2,5]). The analysis of these operators rely on very precise estimates about the uniform asymptotic behaviour of Bessel's functions of different orders and their derivatives. These estimates will be needed, also, when working with discrete Fourier–Neumann series.

In the same way as the Fourier–Neumann series are used to solve dual integral equations, the discrete Fourier–Neumann will help us to solve the dual series equations. In this problem, we must find a sequence $\{r(s)\}_{s\geqslant 0}$ such that

$$\begin{cases} \sum_{s=1}^{\infty} r(s)p_s^\beta J_\alpha(p_sx) = f(x) & \text{if } 0<x\leqslant 1, \\ \sum_{s=1}^{\infty} r(s)J_\alpha(p_sx) = 0 & \text{if } 1<x<a \end{cases}$$

for a given function f. Dual integral equations have applications to certain physical problems in a semi-infinite medium. The corresponding problems in which the medium is confined within a circular cylinder or between a pair of parallel planes can be reduced to dual series equations (see [8, Chapters 2,9,10, 9, Chapter 6]). We will deal with dual series equations in a forthcoming paper.

The aim of this paper is to study the convergence of $S_n b$ in the $l^p(\mathbb{N})$-norm. This involves two problems:

(a) To obtain uniform boundedness of the operator $S_n b$ in $l^p(\mathbb{N})$.
(b) To find the subspace of $l^p(\mathbb{N})$ consisting of the sequences b that can be approximated in the $l^p(\mathbb{N})$-norm by its discrete Fourier–Neumann series, i.e., to describe the space

$$b_{p,\alpha} = \overline{\operatorname{span}}\,\{j_n^\alpha\}_{n\geqslant 0} \quad (\text{closure in } l^p(\mathbb{N})).$$

In order to solve (a) we will decompose S_n in a suitable way which reduces the problem to showing the boundedness, with discrete weights, of some operators that will be compared with the discrete Hilbert transform. Moreover, we will need some bounds for the Bessel functions and some results on discrete A_p weights.

Regarding (b), we define the operator $\mathscr{H}_\alpha$, with $\alpha \geqslant -\frac{1}{2}$, given by

$$\mathscr{H}_\alpha(b,x) = \sum_{s=1}^{\infty} b(s)\,\frac{J_\alpha(p_s x)\sqrt{2x}}{a|J_{\alpha+1}(ap_s)|}, \quad 0<x<a \tag{1}$$

for suitable sequences $\{b(s)\}_{s\geqslant 1}$.

Now, we consider the formula (see [9, Section 6.6, Lemma 1])

$$\begin{aligned}\sum_{s=1}^{\infty} & \frac{2J_{\alpha+\gamma+2n+1}(p_s)J_\alpha(p_s x)}{a^2 p_s^{\gamma+1} J_{\alpha+1}^2(ap_s)} \\ &= \frac{\Gamma(\alpha+n+1)}{2^\gamma \Gamma(\alpha+1)\,\Gamma(n+\gamma+1)}\,x^\alpha(1-x^2)^\gamma \\ &\quad \times {}_2F_1(-n, n+\gamma+\alpha+1; \alpha+1; x^2)\chi_{[0,1]}(x)\end{aligned}$$

which holds for n a positive integer or zero and $\alpha,\gamma > -1$. Taking $\gamma=0$, we can write this last expression, using $\mathscr{H}_\alpha$, as

$$\mathscr{H}_\alpha(j_n^\alpha, x) = \sqrt{2(\alpha+2n+1)}\,x^{\alpha+1/2}P_n^{(\alpha,0)}(1-2x^2)\chi_{[0,1]}, \tag{2}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the nth Jacobi polynomial of order (α,β) for $\alpha,\beta > -1$, and therefore $\operatorname{supp}(\mathscr{H}_\alpha(j_n^\alpha)) \subseteq [0,1]$. So, it is clear that not every sequence can be approximated by its discrete Fourier–Neumann series. We only need to deal with sequences such that $\operatorname{supp}(\mathscr{H}_\alpha) \subseteq [0,1]$. This leads us to consider, in a natural way, the $\chi_{[0,1]}$-multiplier for $\mathscr{H}_\alpha$, i.e., the operator M_α defined by

$$\mathscr{H}_\alpha(M_\alpha b) = \chi_{[0,1]}\mathscr{H}_\alpha b.$$

The paper is organized as follows: In Section 2, we prove the uniform boundedness of the operator S_n; also, all the tools needed for this are introduced. Section 3 contains the results related to M_α and in the last section we identify the spaces $b_{p,\alpha}$.

Throughout this paper, unless otherwise stated, we use C (or C_1) to denote a positive constant independent of n (and all other variables), which can assume different values in different occurrences. Also, in what follows, $a_n \sim b_n$, for $a_n, b_n > 0$, means $C \leqslant a_n/b_n \leqslant C_1$.

2. Uniform boundedness of the partial sums

In order to estimate S_n we will need appropriate bounds for the Bessel functions. From the well-known estimates (see [6, 11, Section 3.1(8), p. 40, Section 7.21(1), p. 199])

$$J_\mu(x) = \frac{x^\mu}{2^\mu \Gamma(\mu+1)} + O(x^{\mu+2}), \quad x \to 0+ \tag{3}$$

and

$$J_\mu(x) = \sqrt{\frac{2}{\pi x}}\left[\cos\left(x - \frac{\mu\pi}{2} - \frac{\pi}{4}\right) + O(x^{-1})\right], \quad x \to \infty, \tag{4}$$

where the O terms depend on μ, we have, for $\mu \geqslant -\frac{1}{2}$,

$$|J_\mu(x)| \leqslant C_\mu x^{-1/2}, \quad x \in (0, \infty). \tag{5}$$

The formula $2J'_\mu(x) = \frac{\alpha}{x} J_\mu(x) - J_{\mu+1}(x)$ proves the same inequality for $J'_\mu(x)$ and $x > \frac{1}{2}$.

Some bounds for J_μ and J'_μ with constants independent of μ are also available. For instance (see [1,5]),

$$|J_\mu(x)| \leqslant C \begin{cases} (\mu - x)^{-1} & \text{if } \mu/2 < x \leqslant \mu - \mu^{1/3}, \\ \mu^{-1/3} & \text{if } \mu - \mu^{1/3} \leqslant x \leqslant \mu + \mu^{1/3}, \\ \mu^{-1/4}(x-\mu)^{-1/4} & \text{if } \mu + \mu^{1/3} \leqslant x \leqslant 2\mu, \\ x^{-1/2} & \text{if } 2\mu \leqslant x, \end{cases} \tag{6}$$

$$|J'_\mu(x)| \leqslant C \begin{cases} \mu^{-1/2}(\mu - x)^{-1/2} & \text{if } \mu/2 < x \leqslant \mu - \mu^{1/3}, \\ \mu^{-2/3} & \text{if } \mu - \mu^{1/3} \leqslant x \leqslant \mu + \mu^{1/3}, \\ \mu^{-3/4}(x-\mu)^{1/4} & \text{if } \mu + \mu^{1/3} \leqslant x \leqslant 2\mu, \\ x^{-1/2} & \text{if } 2\mu \leqslant x. \end{cases} \tag{7}$$

In the interval $0 < x \leqslant \mu/2$, for each $\beta \in \mathbb{R}$ with $\beta + \mu \geqslant 0$ there exists some constant C_β depending only on β, such that

$$|J_\mu(x)| x^\beta \leqslant C_\beta \mu^{\beta - 1/2} \left(\frac{e}{4}\right)^\mu \tag{8}$$

(see [11, Section 3.31, p. 49]). From $2J'_\mu(x) = J_{\mu-1}(x) - J_{\mu+1}(x)$, the same inequality for $J'_\mu(x)$ can be obtained. It is easy to deduce from (6), (7), and (8) that

$$|J_\mu(x)| \leqslant C x^{-1/4}(|x-\mu| + \mu^{1/3})^{-1/4}, \quad x \in (0, \infty), \tag{9}$$

$$|J'_\mu(x)| \leqslant C x^{-3/4}(|x-\mu| + \mu^{1/3})^{1/4}, \quad x \in (0, \infty) \tag{10}$$

with some constant C independent of μ. Bounds as (9) and(10) were used in [3,10].

Some information about the positive zeros of the function J_μ will be needed. Let μ_s be the sth, $s\geqslant 1$, positive zero of J_μ. Then we can show, using (4), that

$$|J_{\mu+1}(\mu_s)|\sim\mu_s^{-1/2}. \tag{11}$$

A very important issue in our reasoning will be to know where μ_s lies. As for this point we have

$$\mu_s=\pi s-\tfrac{1}{2}\pi \quad \text{if } \mu=-\tfrac{1}{2}, \tag{12}$$

$$\mu_s\in\left(s\pi-\tfrac{1}{4}\pi+\tfrac{1}{2}\mu\pi, s\pi-\tfrac{1}{8}\pi+\tfrac{1}{4}\mu\pi\right) \quad \text{if } -\tfrac{1}{2}<\mu\leqslant\tfrac{1}{2} \tag{13}$$

(see [11, Section 15.33, p. 490]) and

$$\mu_s\in\left(s\pi+\frac{1}{2}\mu\pi-\frac{1}{2}\pi, s\pi+\frac{1}{2}\mu\pi-\frac{1}{4}\pi\right) \quad \text{if } \mu>\frac{1}{2} \text{ and } \mu_s>\frac{(2\mu+1)(2\mu+3)}{\pi} \tag{14}$$

(see [11, Section 15.35, p. 492]).

The main result in this section is the following.

Theorem 1. *Let $\alpha\geqslant-\frac{1}{2}$ and $1<p<\infty$. Then there exists a constant C independent of n and b such that*

$$||S_n b||_{l^p(\mathbb{N})}\leqslant C||b||_{l^p(\mathbb{N})}, \quad \forall b\in l^p(\mathbb{N}) \;\Leftrightarrow\; \tfrac{4}{3}<p<4. \tag{15}$$

First of all, we are going to give an appropriate expression for S_n. For a sequence $b\in l^2(\mathbb{N})\cap l^p(\mathbb{N})$, it can be described as

$$S_n(b,j)=\sum_{s=1,s\neq j}^{\infty} b(s)K_n(s,j)+b(j)K_n(j,j),$$

where

$$K_n(s,j)=\sum_{k=0}^{n} j_k^\alpha(s)j_k^\alpha(j).$$

So, using the identity

$$\sum_{k=0}^{n} 2(\alpha+2k+1)J_{\alpha+2k+1}(r)J_{\alpha+2k+1}(t)$$
$$=\begin{cases} \dfrac{rt}{r^2-t^2}[rJ_{\alpha+1}(r)J_\alpha(t)-tJ_\alpha(r)J_{\alpha+1}(t) \\ \quad +rJ'_{\alpha+2n+2}(r)J_{\alpha+2n+2}(t)-tJ_{\alpha+2n+2}(r)J'_{\alpha+2n+2}(t)] & \text{for } r\neq t, \\ r^2[(J'_\alpha(r))^2+\left(1-\dfrac{\alpha^2}{r^2}\right)J_\alpha^2(r) \\ \quad -(J'_{\alpha+2n+2}(r))^2-\left(1-\dfrac{(\alpha+2n+2)^2}{r^2}\right)J^2_{\alpha+2n+2}(r)] & \text{for } r=t, \end{cases}$$

(the case $r \neq t$ can be found in [10] and the case $r = t$ is obtained taking the limit when $t \to r$), we have

$$S_n b = W_1 b - W_2 b + W_3 b + W_{4,n} b - W_{5,n} b - W_{6,n} b,$$

where

$$W_1(b,j) = \frac{2J_\alpha(p_j)}{\sqrt{a}|J_{\alpha+1}(ap_j)|} \sum_{s=1, s\neq j}^{\infty} \frac{ap_s J_{\alpha+1}(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|} \frac{b(s)}{(ap_s)^2 - (ap_j)^2},$$

$$W_2(b,j) = \frac{2ap_j J_{\alpha+1}(p_j)}{\sqrt{a}|J_{\alpha+1}(ap_j)|} \sum_{s=1, s\neq j}^{\infty} \frac{J_\alpha(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|} \frac{b(s)}{(ap_s)^2 - (ap_j)^2},$$

$$W_3(b,j) = 2b(j) \frac{(J'_\alpha(p_j))^2 + (1 - \frac{\alpha^2}{p_j^2}) J_\alpha^2(p_j)}{a^2 (J_{\alpha+1}(ap_j))^2},$$

$$W_{4,n}(b,j) = \frac{2J_v(p_j)}{\sqrt{a}|J_{\alpha+1}(ap_j)|} \sum_{s=1, s\neq j}^{\infty} \frac{ap_s J'_v(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|} \frac{b(s)}{(ap_s)^2 - (ap_j)^2},$$

$$W_{5,n}(b,j) = \frac{2ap_j J'_v(p_j)}{\sqrt{a}|J_{\alpha+1}(ap_j)|} \sum_{s=1, s\neq j}^{\infty} \frac{J_v(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|} \frac{b(s)}{(ap_s)^2 - (ap_j)^2},$$

$$W_{6,n}(b,j) = 2b(j) \frac{(J'_v(p_j))^2 + (1 - \frac{v^2}{p_j^2}) J_v^2(p_j)}{a^2 (J_{\alpha+1}(ap_j))^2}$$

with $v = \alpha + 2n + 2$.

To prove the boundedness of S_n, we will need some results about discrete A_p weights: Let $1 < p < \infty$. A discrete weight w is said to belong to the $A_p(S)$ class, $S \subseteq \mathbb{Z}$, denoted by $w \in A_p(S)$, if

$$\left(\sum_{s\in I} w(s) \right) \left(\sum_{s\in I} w^{-1/(p-1)}(s) \right)^{p-1} \leqslant C \left(\sum_{s\in I} 1 \right)^p,$$

where I is any subset of S and C is independent of I. The weights in $A_p(S)$ characterize the boundedness of the discrete Hilbert transform

$$H(b,j) = \sum_{s\in S, s\neq j} \frac{b(s)}{j-s}.$$

In fact, in [7] it is proved that

$$||Hb||_{l^p(S,w)} \leqslant C ||b||_{l^p(S,w)}, \quad \forall b \in l^p(S,w) \ \Leftrightarrow \ w \in A_p(S). \tag{16}$$

Besides, the norm of the Hilbert transform and the constant in the $A_p(S)$ definition depend only on each other. This allows us to use the uniform $A_p(S)$ theory: Let us suppose that a family of weights $\{w_n\}_{n\geqslant 0}$ satisfies the $A_p(S)$ condition with the same constant C (in this case, we say that $w_n \in A_p(S)$ uniformly). Then the discrete Hilbert transform H is uniformly bounded from $l^p(S, w_n)$ into itself, that is, with a constant independent of n.

Using the following lemma we will estimate the $l^p(\mathbb{N})$-norm of some of the operators involved in S_n.

Lemma 1. *Let* $\alpha \geqslant -\frac{1}{2}$, $1<p<\infty$ *and* $w \in A_p(\mathbb{N})$. *Then,*

(a) *the operator*

$$H_-(b,j) = \sum_{s=1, s\neq j}^{\infty} \frac{b(s)}{ap_j - ap_s}$$

satisfies the inequality

$$\|H_- b\|_{l^p(\mathbb{N},w)} \leqslant C\|b\|_{l^p(\mathbb{N},w)} \quad \forall b \in l^p(\mathbb{N}, w),$$

(b) *the operator*

$$H_+(b,j) = \sum_{s=1, s\neq j}^{\infty} \frac{b(s)}{ap_j + ap_s}$$

satisfies the inequality

$$\|H_+ b\|_{l^p(\mathbb{N},w)} \leqslant C\|b\|_{l^p(\mathbb{N},w)} \quad \forall b \in l^p(\mathbb{N}, w).$$

Moreover, the norm of these operators and the constant in the $A_p(\mathbb{N})$ *definition depend only on each other.*

Proof. (a) The result follows by showing that for $j-\frac{1}{4} \leqslant x \leqslant j+\frac{1}{4}$

$$\left| H_-(b,j) - \frac{1}{\pi} H(b,j) \right| \leqslant C \int_{|x-y|\geqslant 1/2} \frac{|f(y)|}{(x-y)^2}\, dy, \tag{17}$$

where H denotes the discrete Hilbert transform, and by taking the function $f(y) = \sum_{s=1}^{\infty} b(s)\chi_{[s-1/4, s+1/4]}(y)$. In this situation and defining the weight $W(x)$ to be $w(j)$ for $j-\frac{1}{4} \leqslant x \leqslant j+\frac{1}{4}$ and linear in between (so, it is clear that $W \in A_p((0,\infty))$), we can conclude that

$$\begin{aligned}
&\|H_- b\|^p_{l^p(\mathbb{N},w)} \\
&\quad = \sum_{j=1}^{\infty} |H_-(b,j)|^p w(j) \\
&\quad \leqslant C\left(\sum_{j=1}^{\infty} |H(b,j)|^p w(j) + \sum_{j=1}^{\infty} \int_{j-1/4}^{j+1/4} \left(\int_{|x-y|\geqslant 1/2} \frac{|f(y)|}{(x-y)^2}\, dy \right)^p W(x)\, dx \right) \\
&\quad \leqslant C\left(\|Hb\|^p_{l^p(\mathbb{N},w)} + \left\| \int_{|x-y|\geqslant 1/2} \frac{|f(y)|}{(x-y)^2}\, dy \right\|^p_{L^p((0,\infty),W)} \right).
\end{aligned}$$

Clearly

$$\int_{|x-y|\geqslant 1/2}\frac{|f(y)|}{(x-y)^2}\,dy\leqslant CM(f,x),$$

where M is the Hardy–Littlewood maximal operator. It is well known that M satisfies

$$||Mf||_{L^p((0,\infty),W)}\leqslant C||f||_{L^p((0,\infty),W)} \;\Leftrightarrow\; W\in A_p((0,\infty)),$$

and the constant C only depends on the constant in the A_p definition. From this and (16) we get the estimate

$$||H_-b||_{L^p(w)}\leqslant C(||b||_{l^p(\mathbb{N},w)}+||f||_{L^p((0,\infty),W)})\leqslant C||b||_{l^p(\mathbb{N},w)}.$$

Now, we must prove (17). From (12)–(14) we have $|ap_j-ap_s-\pi(j-s)|\leqslant C$ and $|ap_j-ap_s|\sim|j-s|$. So,

$$\left|H_-(b,j)-\frac{1}{\pi}H(b,j)\right|\leqslant C\sum_{s=1,s\neq j}^{\infty}\frac{|b(s)|}{(j-s)^2}.$$

Taking into account that $|b(s)|=2\int_{s-1/4}^{s+1/4}|f(y)|\,dy$ and $|j-s|\sim|x-y|$ for $s-\frac{1}{4}\leqslant y\leqslant s+\frac{1}{4}$ and $j-\frac{1}{4}\leqslant x\leqslant j+\frac{1}{4}$, with $s\neq j$, we conclude

$$\begin{aligned}\left|H_-(b,j)-\frac{1}{\pi}H(b,j)\right|&\leqslant C\sum_{s=1,s\neq j}^{\infty}\frac{1}{(j-s)^2}\int_{s-1/4}^{s+1/4}|f(y)|\,dy\\&\sim\sum_{s=1,s\neq j}^{\infty}\int_{s-1/4}^{s+1/4}\frac{|f(y)|}{(x-y)^2}\,dy\leqslant C\int_{|x-y|\geqslant 1/2}\frac{|f(y)|}{(x-y)^2}\,dy.\end{aligned}$$

(b) Taking into account that $ap_s\sim s$ it is easily verified that

$$|H_+(b,j)|\sim\sum_{s=1,s\neq j}^{\infty}\frac{|b(s)|}{j+s}.$$

Now, considering the sequence $c(s)=|b(s)|$, for $s\geqslant 1$, and $c(s)=0$, for $s\leqslant 0$, and the weight $\bar{w}(s)=w(|s|)$ and $\bar{w}(0)=0$, we have

$$||H_+(b,j)||_{L^p(\mathbb{N},w)}\leqslant C\left|\left|\left\{\sum_{s\in\mathbb{Z},-s\neq j}^{\infty}\frac{c(s)}{j+s}\right\}_{j\in\mathbb{Z}}\right|\right|_{l^p(\mathbb{Z},\bar{w})}=C||H(d)||_{l^p(\mathbb{Z},\bar{w})},$$

where $d(s)=c(-s)$. So, the desired inequality can be obtained from (16) for $S=\mathbb{Z}$. □

The following lemma, which shows that some weights are in $A_p(\mathbb{N})$, will be used later in connection with the previous one:

Lemma 2. *Let $\alpha \geqslant -\frac{1}{2}$ and $1<p<\infty$. Then*

(a) *if $-1<\gamma<p-1$, $\{p_s^\gamma\}_{s\geqslant 1}\in A_p(\mathbb{N})$.*
(b) *if $\frac{4}{3}<p<4$, $\{p_s^{p/4}||p_s-v|+v^{1/3}|^{-p/4}\}_{s\geqslant 1}\in A_p(\mathbb{N})$ uniformly in v.*

Proof. (a) This part is obvious using (12)–(14) and the equivalence

$$\sum_{s=1}^{k} s^{\varepsilon} \sim \begin{cases} k^{\varepsilon+1} & \text{if } \varepsilon > -1, \\ \log(k+1) & \text{if } \varepsilon = -1, \\ 1 & \text{if } \varepsilon < -1. \end{cases} \tag{18}$$

(b) It is clear that, for $p^{-1}+q^{-1}=1$,

$$w\in A_p(\mathbb{N}) \;\Leftrightarrow\; w^{-q/p}\in A_q(\mathbb{N}).$$

So, we will check that $\{p_s^{-q/4}||p_s-v|+v^{1/3}|^{q/4}\}_{s\geqslant 1}\in A_q(\mathbb{N})$, if $\frac{4}{3}<q<4$. Using the equivalence $p_s^{-q/4}||p_s-v|+v^{1/3}|^{q/4}\sim p_s^{-q/4}(|p_s-v|^{q/4}+v^{q/12})$, it will be enough to prove that $\{p_s^{-q/4}\}_{s\geqslant 1}\in A_q(\mathbb{N})$ and $\{p_s^{-q/4}|p_s-v|^{q/4}\}_{s\geqslant 1}\in A_q(\mathbb{N})$ uniformly in v. Using (a), we have $\{p_s^{-q/4}\}_{s\geqslant 1}\in A_q(\mathbb{N})$ for $q<4$. Now, showing that

$$\begin{cases} -1<u<q-1, \\ -1<v<q-1, \\ -1<u+w<q-1 \end{cases} \Rightarrow \{s^u|s-\delta|^v\}_{s\geqslant 1} \text{in } A_p(\mathbb{N}) \text{uniformly in } \delta \tag{19}$$

and using (12)–(14), $\{p_s^{-q/4}|p_s-v|^{q/4}\}_{s\geqslant 1}\in A_q(\mathbb{N})$ uniformly in v, if $\frac{4}{3}<q<4$. Taking into account (18) and the behaviour of the weight in the intervals $[1,[\delta/2])$, $[[\delta/2],[2\delta])$ and $[[2\delta],\infty)$, where $[m]$ denotes the integer part of m, we can prove (19). □

Proof (Proof of Theorem 1). First, we are going to prove that $\frac{4}{3}<p<4$ is a necessary condition for the uniform boundedness of S_n.

It is clear that the operator

$$T_n b = S_n b - S_{n-1} b = c_n(b) j_n^\alpha$$

must be bounded. Then, using duality, this fact implies that

$$||j_n^\alpha||_{l^p(\mathbb{N})}||j_n^\alpha||_{l^q(\mathbb{N})}\leqslant C. \tag{20}$$

From (5) and taking into account the behaviour of p_s, we can prove that $j_n^\alpha\in l^p(\mathbb{N})$ if $1<p<\infty$. In that case, asymptotic estimates for $J_{\alpha+2n+1}$ allow us to show that

$$||j_n^\alpha||_{l^p(\mathbb{N})} \sim \begin{cases} n^{\frac{1}{p}-\frac{1}{2}} & \text{if } p<4, \\ n^{-\frac{1}{4}}(\log n)^{\frac{1}{4}} & \text{if } p=4, \\ n^{\frac{1}{3p}-\frac{1}{3}} & \text{if } p>4. \end{cases}$$

Then, this and (20) give the necessary condition $\frac{4}{3}<p<4$.

Now, let us suppose that $\frac{4}{3}<p<4$. We prove the uniform boundedness of $S_n b$. We start analyzing W_1. Using $\frac{2r}{r^2-t^2}=\frac{1}{r-t}+\frac{1}{r+t}$, we can write

$$W_1 b = -W_1^1 b + W_1^2 b,$$

where

$$W_1^1(b,j)=\frac{J_\alpha(p_j)}{\sqrt{a}|J_{\alpha+1}(ap_j)|}H_-\left(\frac{J_{\alpha+1}(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|}b(s),j\right)$$

and

$$W_1^2(b,j)=\frac{J_\alpha(p_j)}{\sqrt{a}|J_{\alpha+1}(ap_j)|}H_+\left(\frac{J_{\alpha+1}(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|}b(s),j\right).$$

From (5), (11), (a) in Lemma 1 and (a) in Lemma 2 it can be concluded that, for $1<p<\infty$,

$$\begin{aligned}||W_1^1 b||_{l^p(\mathbb{N})} &\leqslant C\left\|\left\{H_-\left(\frac{J_{\alpha+1}(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|}b(s),j\right)\right\}_{j\geqslant 1}\right\|_{l^p(\mathbb{N})}\\ &\leqslant C\left\|\left\{\left|\frac{J_{\alpha+1}(p_j)}{\sqrt{a}J_{\alpha+1}(ap_j)}\right||b(j)|\right\}_{j\geqslant 1}\right\|_{l^p(\mathbb{N})}\leqslant C||b||_{l^p(\mathbb{N})}.\end{aligned}$$

In a similar way, but considering (b) in Lemma 1, it is possible to deduce that, if $1<p<\infty$,

$$\begin{aligned}||W_1^2 b||_{l^p(\mathbb{N})} &\leqslant C\left\|\left\{H_+\left(\frac{J_{\alpha+1}(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|}b(s),j\right)\right\}_{j\geqslant 1}\right\|_{l^p(\mathbb{N})}\\ &\leqslant C\left\|\left\{\left|\frac{J_{\alpha+1}(p_j)}{\sqrt{a}J_{\alpha+1}(ap_j)}\right||b(j)|\right\}_{j\geqslant 1}\right\|_{l^p(\mathbb{N})}\leqslant C||b||_{l^p(\mathbb{N})}.\end{aligned}$$

The operator W_2 works like W_1. The boundedness of W_3 follows from (5), (11) and the bound $|J'_\alpha(x)|\leqslant C_\alpha x^{-1/2}$, which holds for $x>\frac{1}{2}$; in fact, we have $|W_3(b,j)|\leqslant C|b(j)|$.

Now, we will check that $W_{4,n}$ is uniformly bounded in $l^p(\mathbb{N})$ for $\frac{4}{3}<p<4$. Using this and taking into account that the adjoint operator of $W_{4,n}$ is $-W_{5,n}$, we will obtain that $W_{5,n}$ also satisfies the desired inequality in the same interval of values of p.

As for W_1, we can write

$$W_{4,n}b = -W_{4,n}^1 b + W_{4,n}^2 b,$$

where

$$W_{4,n}^1(b,j)=\frac{J_\nu(p_j)}{2\sqrt{a}|J_{\alpha+1}(ap_j)|}H_-\left(\frac{J'_\nu(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|}b(s),j\right)$$

and

$$W^2_{4,n}(b,j) = \frac{J_v(p_j)}{2\sqrt{a}|J_{\alpha+1}(ap_j)|} H_+\left(\frac{J'_v(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|} b(s), j\right).$$

In this way, using consecutively (9), (11), (a) in Lemma 1, (b) in Lemma 2, (10) and, again, (11) we have

$$\begin{aligned} &||W^1_{4,n}b||_{l^p(\mathbb{N})} \\ &\leqslant C\left|\left|\left\{H_-\left(\frac{J'_v(p_s)}{\sqrt{a}|J_{\alpha+1}(ap_s)|} b(s), j\right)\right\}_{j\geqslant 1}\right|\right|_{l^p(\mathbb{N},\{p_j^{p/4}||p_j-v|+v^{1/3}|^{-p/4}\}_{j\geqslant 1})} \\ &\leqslant C\left|\left|\left\{\left|\frac{J'_v(p_j)}{\sqrt{a}J_{\alpha+1}(ap_j)}\right||b(j)|\right\}_{j\geqslant 1}\right|\right|_{l^p(\mathbb{N},\{p_j^{p/4}||p_j-v|+v^{1/3}|^{-p/4}\}_{j\geqslant 1})} \\ &\leqslant C||b||_{l^p(\mathbb{N})}. \end{aligned}$$

By (b) in Lemma 1, we can deduce the estimate for $W^2_{4,n}$.

Now, estimates (6)–(8) allow us to conclude that $|W_{6,n}(b,j)| \leqslant C|b(j)|$. So, we can prove the uniform boundedness of $W_{6,n}$, ending the proof. □

3. The $\chi_{[0,1]}$-multiplier for the operator $\mathscr{H}_\alpha$

It is well known that the sequence of functions $\{\frac{J_\alpha(p_s x)\sqrt{2x}}{a|J_{\alpha+1}(ap_s)|}\}_{s\geqslant 1}$ forms a complete orthonormal system in $L^2([0,a], dx)$, usually called Bessel system. Now, we define the operator $\mathscr{J}_\alpha$, for $\alpha \geqslant -\frac{1}{2}$, by

$$\mathscr{J}_\alpha(f,j) = \int_0^a f(x) \frac{J_\alpha(p_j x)\sqrt{2x}}{a|J_{\alpha+1}(ap_j)|} dx, \quad j = 1, 2, \ldots$$

for suitable functions f. The operator $\mathscr{J}_\alpha$ gives the Fourier coefficients associated with the Bessel system and its inverse operator is $\mathscr{H}_\alpha$. So, it is not difficult to check that

$$||\mathscr{J}_\alpha f||_{l^2(\mathbb{N})} = ||f||_{L^2([0,a],dx)} \quad \text{and} \quad \mathscr{H}_\alpha(\mathscr{J}_\alpha f) = f$$

for $f \in L^2([0,a], dx)$. Moreover, for $b \in l^2(\mathbb{N})$, we have

$$||\mathscr{H}_\alpha b||_{L^2([0,a],dx)} = ||b||_{l^2(\mathbb{N})} \quad \text{and} \quad \mathscr{J}_\alpha(\mathscr{H}_\alpha b) = b.$$

The following relation between $\mathscr{H}_\alpha$ and $\mathscr{J}_\alpha$ can be proved in an easy way:

$$\sum_{s=1}^{\infty} b(s)\mathscr{J}_\alpha(f,j) = \int_0^a \mathscr{H}_\alpha(b,x)f(x)\, dx \tag{21}$$

for $f \in L^2([0,a], dx)$ and $b \in l^2(\mathbb{N})$.

Using $\mathscr{J}_\alpha$, the operator M_α can be described, at least for $b \in l^2(\mathbb{N}) \cap l^p(\mathbb{N})$, as

$$M_\alpha b = \mathscr{J}_\alpha(\chi_{[0,1]}\mathscr{H}_\alpha b). \tag{22}$$

This description will be very useful in the proof of the following result.

Theorem 2. *Let $\alpha \geqslant -\frac{1}{2}$ and $1 < p < \infty$. Then there exists a constant C such that*

$$||M_\alpha b||_{l^p(\mathbb{N})} \leqslant C||b||_{l^p(\mathbb{N})} \quad \forall b \in l^2(\mathbb{N}) \cap l^p(\mathbb{N}).$$

Therefore, M_α can be extended to an operator (also denoted M_α) bounded from $l^p(\mathbb{N})$ into itself such that

(a) $\mathscr{H}_\alpha(M_\alpha b) = \chi_{[0,1]}\mathscr{H}_\alpha b$, $\forall b \in l^2(\mathbb{N}) \cap l^p(\mathbb{N})$,
(b) $M_\alpha^2 b = M_\alpha b$, $\forall b \in l^p(\mathbb{N})$,
(c) *for $b \in l^p(\mathbb{N})$ and $c \in l^q(\mathbb{N})$, $p^{-1} + q^{-1} = 1$, we have*

$$\sum_{s=1}^{\infty} b(s) M_\alpha(c,s) = \sum_{s=1}^{\infty} M_\alpha(b,s) c(s).$$

Proof. From (22), we obtain

$$M_\alpha(b,j) = \sum_{s=1, s\neq j}^{\infty} \frac{2b(s)}{a^2|J_{\alpha+1}(ap_s)||J_{\alpha+1}(ap_j)|} \int_0^1 J_\alpha(p_s x) J_\alpha(p_j x) x\,dx + \frac{2b(j)}{a^2(J_{\alpha+1}(ap_j))^2} \int_0^1 (J_\alpha(p_j x))^2 x\,dx.$$

Lommel's formula

$$\int_0^1 J_\alpha(rx) J_\alpha(tx) x\,dx = \begin{cases} \dfrac{1}{r^2 - t^2}(rJ_{\alpha+1}(r)J_\alpha(t) - tJ_\alpha(r)J_{\alpha+1}(t)) & \text{if } r \neq t, \\ (J_\alpha'(r))^2 + \left(1 - \dfrac{\alpha^2}{r^2}\right) J_\alpha^2(r) & \text{if } r = t \end{cases}$$

leads us to

$$M_\alpha b = W_1 b - W_2 b + W_3 b,$$

where W_1, W_2 and W_3, are bounded operators as we saw in the proof of Theorem 1.

Now, taking into account that $\mathscr{J}_\alpha(\mathscr{H}_\alpha b) = b$ and using standard density arguments, statements (a) and (b) follow easily. So, let us prove (c).

Let U_1 and U_2 be the bilinear operators on $l^p(\mathbb{N}) \times l^q(\mathbb{N})$ given by

$$U_1(b,c) = \sum_{s=1}^{\infty} b(s) M_\alpha(c,s)$$

and

$$U_2(b,c) = \sum_{s=1}^{\infty} M_\alpha(b,s) c(s).$$

U_1 and U_2 are well defined by Hölder inequality. So, if $U_1 = U_2$, for all $(b,c)\in(l^2(\mathbb{N})\times l^2(\mathbb{N}))\cap(l^p(\mathbb{N})\times l^q(\mathbb{N}))$, which is a dense subset of $l^p(\mathbb{N})\times l^q(\mathbb{N})$, we can obtain (c). Using (22) and (21) twice we finish the proof:

$$U_1(b,c) = \sum_{s=1}^{\infty} b(s)\mathscr{I}(\chi_{[0,1]}\mathscr{H}_\alpha c, s) = \int_0^a \chi_{[0,1]}(x)\mathscr{H}_\alpha(b,x)\mathscr{H}_\alpha(c,x)\,dx$$
$$= \sum_{s=1}^{\infty} \mathscr{I}_\alpha(\chi_{[0,1]}\mathscr{H}_\alpha b, s)c(s) = U_2(b,c). \qquad \square$$

4. The $e_{p,\alpha}$ spaces

As a standard consequence of Theorem 1, we obtain that $S_n b\to b$, in $l^p(\mathbb{N})$, for all $b\in b_{p,\alpha}$, if $\frac{4}{3}<p<4$ and $\alpha\geqslant-\frac{1}{2}$. This will be more interesting if we can describe the spaces $b_{p,\alpha}$. We do it in this section.

From Theorem 2, for $\alpha\geqslant-\frac{1}{2}$ and $1<p<\infty$ we can define the spaces

$$e_{p,\alpha} = \{b\in l^p(\mathbb{N}) : M_\alpha b = b\} = M_\alpha(l^p(\mathbb{N})),$$

endowed with the topology induced by $l^p(\mathbb{N})$.

It is clear, using $l^{p_1}(\mathbb{N})\subset l^{p_2}(\mathbb{N})$, that $e_{p_1,\alpha}\subset e_{p_2,\alpha}$ for $1<p_1<p_2<\infty$. In the next proposition we will prove that the dual space $(e_{p,\alpha})^*$ is isomorphic to $e_{q,\alpha}$, $p^{-1}+q^{-1}=1$, in the standard sense.

Proposition 1. *Let $\alpha\geqslant-\frac{1}{2}$, $1<p<\infty$ and let T be a bounded linear operator on $e_{p,\alpha}$ into $\mathbb{R}$. Then, there exists a unique sequence $c\in e_{q,\alpha}$, $p^{-1}+q^{-1}=1$, such that*

$$T(b) = \sum_{s=1}^{\infty} b(s)c(s) \quad \forall b\in e_{p,\alpha}. \tag{23}$$

Furthermore $\|T\|\sim\|c\|_{l^q(\mathbb{N})}$.

Proof. Let us take $T\in(e_{p,\alpha})^*$. By the Hahn–Banach theorem, T can be extended to $T\in(l^p(\mathbb{N}))^*$ preserving its norm $\|T\|$. By duality, there exists $d\in l^q(\mathbb{N})$ such that

$$T(b) = \sum_{s=1}^{\infty} b(s)d(s) \quad \forall b\in l^p(\mathbb{N})$$

and $\|T\| = \|d\|_{l^p(\mathbb{N})}$. Taking $c = M_\alpha d$, from (b) in Theorem 2, it follows that $c\in e_{q,\alpha}$. Let see that this sequence is what we need. For $b\in e_{p,\alpha}$, from (c) in Theorem 2, we have

$$T(b) = \sum_{s=1}^{\infty} b(s)d(s) = \sum_{s=1}^{\infty} M_\alpha(b,s)d(s) = \sum_{s=1}^{\infty} b(s)M_\alpha(d,s) = \sum_{s=1}^{\infty} b(s)c(s).$$

By Hölder inequality we obtain $\|T\|\leqslant\|M_\alpha d\|_{l^q(\mathbb{N})}$, thus the equivalence $\|T\|\sim\|c\|_{l^q(\mathbb{N})}$ follows immediately.

To prove the uniqueness, we assume that $p \geqslant 2$ and that there exists c and c' in $e_{p,\alpha}$ satisfying (23). Then

$$\sum_{s=1}^{\infty} b(s)(c(s) - c'(s)) = 0 \quad \forall b \in l^p(\mathbb{N}).$$

Taking $b = c - c' \in e_{q,\alpha} \subset e_{p,\alpha}$ it follows that $c - c' = 0$. The case $p < 2$ is a simple consequence of this considering that $l^q(\mathbb{N})$ is reflexive and, since $e_{q,\alpha}$ is closed, then $e_{q,\alpha}$ is also reflexive. $\square$

The following result characterizes the spaces $b_{p,\alpha}$.

Theorem 3. *Let* $\alpha \geqslant -\frac{1}{2}$ *and* $\frac{4}{3} < p$. *Then* $b_{p,\alpha} = e_{p,\alpha}$.

Proof. *Case* $p = 2$: The spaces $b_{2,\alpha}$ and $e_{2,\alpha}$ are well defined. Also $M_\alpha(j_n^\alpha) = j_n^\alpha$. In other words, $b_{2,\alpha} \subseteq e_{2,\alpha}$. If they were not equal, by the Hahn–Banach theorem, there should exits some $T \in (e_{2,\alpha})^*$, $T \neq 0$, such that $T(j_n^\alpha) = 0$, $\forall n$. But $(e_{2,\alpha})^* = e_{2,\alpha}$, so there exists a sequence $\phi \in e_{2,\alpha}$, $\phi \neq 0$, such that $\sum_{s=1}^{\infty} \phi(s) j_n^\alpha(s) = 0$, $\forall n$. Then

$$\begin{aligned} 0 &= \sum_{s=1}^{\infty} \phi(s) j_n^\alpha(s) = \sum_{s=1}^{\infty} M_\alpha(\phi, s) j_n^\alpha(s) = \int_0^1 \mathscr{H}_\alpha(\phi, x) \mathscr{H}_\alpha(j_n^\alpha, x)\, dx \\ &= \sqrt{2(\alpha + 2n + 1)} \int_0^1 \mathscr{H}_\alpha(\phi, x) x^{\alpha+1/2} P_n^{(\alpha,0)}(1 - 2x^2)\, dx \end{aligned}$$

for every nonnegative integer n. Now, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are a complete orthogonal system with respect to the measure $(1-x)^\alpha (1+x)^\beta\, dx$. A change of variable proves that the functions $x^{\alpha+1/2} P_n^{(\alpha,0)}(1-2x^2)$ are a complete orthogonal system with respect to the Lebesgue measure on $(0,1)$. Thus, $\mathscr{H}_\alpha \phi = 0$ on $(0,1)$. Since $\phi \in e_{2,\alpha}$, we also have $\mathscr{H}_\alpha \phi = 0$ on (1,a). Therefore, $\mathscr{H}_\alpha \phi = 0$ and we get the contradiction $\phi = 0$.

Case $p > 2$: By the preceding case, we have $j_n^\alpha \in e_{2,\alpha} \subset e_{p,\alpha}$. Thus, $b_{p,\alpha} \subseteq e_{p,\alpha}$. Now, let $b \in e_{p,\alpha}$. Given $\varepsilon > 0$, there exists a sequence $c \in l^2(\mathbb{N}) \cap l^p(\mathbb{N})$ such that $||b - c||_{l^p(\mathbb{N})} \leqslant \varepsilon$. Let $d = M_\alpha c$; then $d \in l^2(\mathbb{N}) \cap l^p(\mathbb{N})$ and $M_\alpha d = d$, so that $d \in e_{2,\alpha} \cap e_{p,\alpha} = b_{2,\alpha} \cap e_{p,\alpha}$. Since M_α is continuous, $||b - d||_{l^p(\mathbb{N})} = ||M_\alpha(b - c)||_{l^p(\mathbb{N})} \leqslant C\varepsilon$. As $d \in b_{2,\alpha}$, there exists $d' \in \operatorname{span}\{j_n^\alpha\}_{n \geqslant 0}$ such that $||d - d'||_{l^2(\mathbb{N})} \leqslant \varepsilon$. Now, the inclusion $e_{2,\alpha} \subset e_{p,\alpha}$ gives $||d - d'||_{L^p(\mathbb{N})} \leqslant C\varepsilon$, so that, by the triangle inequality, $||b - d'||_{L^p(\mathbb{N})} \leqslant C\varepsilon$. This gives the inclusion $e_{p,\alpha} \subseteq b_{p,\alpha}$.

Case $p < 2$: Again it is clear that $j_n^\alpha \in e_{p,\alpha}$ and then $b_{p,\alpha} \subseteq e_{p,\alpha}$. The other inclusion follows if we prove that the only operator $T \in (e_{p,\alpha})^*$ such that $T(b) = 0$ for all $b \in b_{p,\alpha}$ is $T = 0$. For such an operator, we have, in particular, $T(j_n^\alpha) = 0$ for all $n \geqslant 0$. On other hand, by the duality $(e_{p,\alpha})^* = e_{q,\alpha}$, $p^{-1} + q^{-1} = 1$, there exists $\phi \in e_{q,\alpha}$ such that

$$T(b) = \sum_{s=1}^{\infty} b(s)\phi(s) \quad \forall b \in e_{p,\alpha}$$

and

$$\sum_{s=1}^{\infty} j_n^{\alpha}(s)\phi(s) = 0 \quad \forall n \geqslant 0. \tag{24}$$

Now, using the preceding case, Theorem 1 and (24), we conclude that $\phi = 0$. □

The previous theorem allow us to obtain the following corollary.

Corollary 1. *Let* $\alpha \geqslant -\frac{1}{2}$ *and* $\frac{4}{3} < p < 4$. *Then* $\lim_{n\to\infty} S_n b = M_\alpha b$, *in* $l^p(\mathbb{N})$, $\forall b \in l^p(\mathbb{N})$.

Proof. Let us consider $b \in l^p(\mathbb{N})$. Then $M_\alpha b \in e_{p,\alpha}$. Now, by Theorems 1 and 3 $S_n(M_\alpha b) \to M_\alpha b$, in $l^p(\mathbb{N})$. So, we only need to show that $S_n(M_\alpha b) = S_n b$, and this is clear because, by (c) in Theorem 2,

$$\begin{aligned} c_n(M_\alpha b) &= \sum_{s=1}^{\infty} M_\alpha(b,s) j_n^{\alpha}(s) = \sum_{s=1}^{\infty} b(s) M_\alpha(j_n^{\alpha}, s) \\ &= \sum_{s=1}^{\infty} b(s) j_n^{\alpha}(s) = c_n(b). \end{aligned}$$ □

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